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Scaling representation for critical phenomena

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Abstract. A renormalization group method is applied to obtain a representation for the singular part of the free energy of systems described by scaling operators which exhibit a critical phase transition.

The temperature dependence of thermodynamic functions near the critical temperature T_c for a phase transition has been generally assumed to be dominated by a singularity of the form $|T-T_c|^{\mu}$, where the critical exponent μ is real. Experiments on many systems which exhibit critical phenomena, eg liquid–gas transitions, ferromagnetic transitions, etc, support this assumption which has also a theoretical foundation in the scaling arguments of Widom (1965), Domb and Hunter (1965), Patashinskii and Pokrovskii (1965), and Kadanoff (1966). In particular, Wilson (1971, 1972) has deduced this power law dependence from renormalization group arguments showing the universality of the critical exponents, and he obtained an asymptotic expansion in dimensionality for these exponents in collaboration with Fisher (Wilson and Fisher 1972) and Brezin and Wallace (Brezin *et al* 1972). A renormalization group method based more directly on Kadanoff's ideas has been developed by Niemeijer and van Leeuwen (1973, 1974) to calculate the critical exponents and the critical temperature for Ising spin models, and their results for triangular lattices in two dimensions are in excellent agreement with known values. Recently, this approach has been extended by Nauenberg and Nienhuis (1974a, b) to obtain the complete free energy of general Ising spin models, and it can be shown to apply also to models described by a complete set of scaling operators as introduced by Wegner (1972). In this paper we apply this renormalization group approach to derive a representation for the singular part of the free energy which satisfies the homogeneous scaling equation, and obtain explicitly its singularities at the critical temperature.

It has been shown (Nauenberg and Nienhuis 1974a, b, Nauenberg 1975) that the free energy $f(K)$ as a function of the Ising spin interaction variables K_α, or the corresponding variables associated with scaling operators, can be obtained in terms of the self-energy $Lg(K)$ of each Kadanoff cell by solving the scaling equation

$$f(K') = L(f(K)-g(K)) \tag{1}$$

where L is the number of sites in a Kadanoff cell, and K'_α are the cell spin interaction variables determined by the nonlinear renormalization transformation

$$K'_\alpha = F_\alpha(K). \tag{2}$$

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The details for calculating the functions $g(K)$ and $F_\alpha(K)$ for Ising spin systems have been discussed by Niemejer and van Leeuwen (1973, 1974) and by Nauenberg and Nienhuis (1974a, b). However, for our present purpose we require only the generally accepted properties of the renormalization transformations, (equation (2)): (*a*) the existence of an unstable fixed point at K_α^*; and (*b*) the analyticity of $F_\alpha(K)$ and $g(K)$ at this fixed point. For simplicity we assume that the matrix $\partial F_\alpha/\partial K_\beta$ at K^* has only two eigenvalues λ_1 and λ_2 greater than one corresponding to two relevant operators, say temperature and magnetic field, while all the remaining eigenvalues are less than one. Associated with each eigenvalue λ_i we introduce a new variable ζ_i through a nonlinear transformation (Wegner 1972, Nauenberg and Nienhuis 1974a, b)

$$K_\alpha = G_\alpha(\zeta_1, \zeta_2 \ldots) \tag{3}$$

in such a way that the renormalization transformation equation (2), in the ζ_i variables becomes simply

$$\zeta_i' = \lambda_i \zeta_i \tag{4}$$

and $\zeta_i = 0$ corresponds to the fixed point.

The solution of the scaling equation can then be written in the form (Nauenberg and Nienhuis 1974a, b)

$$f(\zeta_1, \zeta_2 \ldots) = \sum_{n=0}^{\infty} \frac{1}{L^n} g(\lambda_1^n \zeta_1, \lambda_2^n \zeta_2 \ldots) + h(\zeta_1, \zeta_2 \ldots) \tag{5}$$

where

$$h(\zeta_1, \zeta_2 \ldots) = \lim_{n\to\infty} \frac{1}{L^n} f(\lambda_1^n \zeta_1, \lambda_2^n \zeta_2 \ldots) \tag{6}$$

satisfies a homogeneous scaling equation corresponding to equation (1) with $g(K) = 0$. Since we assume $\lambda_i > 1$ for $i = 1, 2$, there is a smallest integer m_i such that $\lambda_i^{m_i} \geqslant L$. Hence the m_ith derivative of $f(\zeta_1, \zeta_2 \ldots)$ with respect to ζ_i for $i = 1$ or 2, is given by

$$f^{(m_i)}(\zeta_1, \zeta_2 \ldots) = \sum_{n=0}^{\infty} y_i^n g^{(m_i)}(\lambda_1^n \zeta_1, \lambda_2^n \zeta_2 \ldots) + h^{(m_i)}(\zeta_1, \zeta_2 \ldots) \tag{7}$$

where $y_i = \lambda_i^{m_i}/L > 1$, and therefore it becomes infinite for $\zeta_i = 0$. For our subsequent discussion we set the variables ζ_j corresponding to the irrelevant scaling operators equal to zero, ie $\zeta_j = 0$, $j \neq 1, 2$. Then the m_ith derivative with respect to ζ_i of the regular part $f_r(\zeta_1, \zeta_2)$ of the free energy (equation (7)) is obtained by solving the scaling equation (equation (1)) by an expansion in powers of ζ_1, ζ_2 (Nauenberg and Nienhuis 1974a, b), and can be written in the form

$$f_r^{(m_i)}(\zeta_1, \zeta_2) = -\sum_{n=1}^{\infty} y_i^{-n} g^{(m_i)}(\lambda_1^{-n} \zeta_1, \lambda_2^{-n} \zeta_2). \tag{8}$$

Subtracting this expression from equation (7) we obtain the m_ith derivative of the singular part $f_s(\zeta_1, \zeta_2)$ of the free energy†

$$f_s^{(m_i)}(\zeta_1, \zeta_2) = \sum_{n=-\infty}^{\infty} y_i^n g^{(m_i)}(\lambda_1^n \zeta_1, \lambda_2^n \zeta_2) + h^{(m_i)}(\zeta_1, \zeta_2) \tag{9}$$

† A similar expression has been obtained by J M J van Leeuwen, private communication.

which satisfies the homogeneous scaling equation

$$f_s^{(m_i)}(\lambda_1\zeta_1, \lambda_2\zeta_2) = y_i^{-1} f_s^{(m_i)}(\zeta_1, \zeta_2). \tag{10}$$

Setting

$$f_s^{(m_i)}(\zeta_1, \zeta_2) = \zeta_i^{-\alpha_i} C_i(\zeta_1, \zeta_2) \qquad i = 1, 2 \tag{11}$$

for $\zeta_i > 0$ where $\alpha_i = \ln y_i/\ln \lambda_i$, we find that

$$C_i(\lambda_1\zeta_1, \lambda_2\zeta_2) = C_i(\zeta_1, \zeta_2). \tag{12}$$

Therefore, we can expand $C_i(\zeta_1, \zeta_2)$ in a Fourier series in the variables $\ln \zeta_i/\ln \lambda_i$. For example, keeping the scale invariant variable $\zeta = \zeta_2\zeta_1^{-\Delta}$ fixed, where $\Delta = \ln \lambda_2/\ln \lambda_1$, we have

$$C_1(\zeta_1, \zeta_2) = \sum_{n=-\infty}^{\infty} C_n(\zeta) \exp(2\pi i n \ln \zeta_1/\ln \lambda_1) \tag{13}$$

where the Fourier coefficients $C_n(\zeta)$ are given by

$$\begin{aligned} C_n(\zeta) = {} & \frac{1}{\ln \lambda_1} \int_0^\infty d\zeta_1 \zeta_1^{(\alpha_1-1)} g^{(m_i)}(\zeta_1, \zeta\zeta_1^\Delta) \exp\left(-2\pi i n \frac{\ln \zeta_1}{\ln \lambda_1}\right) \\ & + \frac{1}{\ln \lambda_1} \int_1^{\lambda_1} d\zeta_1 \zeta_1^{(\alpha_1-1)} h^{(m_i)}(\zeta_1, \zeta\zeta_1^\Delta) \exp\left(-2\pi i n \frac{\ln \zeta_1}{\ln \lambda_1}\right). \end{aligned} \tag{14}$$

Similar expressions are obtained for $\zeta_1 < 0$ if we replace ζ_1 by $(-\zeta_1)$ and integrate over the corresponding domain for negative values of ζ_1. Our scaling representation for the singular part of the free energy (equations (11), (13) and (14)) differs from the familiar form first obtained by Widom (1965), Domb and Hunter (1965) and Patashinskii and Pokrovskii (1965) in the occurrence of a periodic dependence† in the variable $\ln \zeta_1/\ln \lambda_1$. In the usual derivations of the scaling representation of $f_s(\zeta_1, \zeta_2)$ based on the homogeneous scaling equation (equation (10)) this periodic dependence is not obtained, ie $C_n = 0$, $n \neq 0$ because of one of the following two additional assumptions: (*a*) that λ_i and correspondingly $L(\lambda_i)$ can be varied continuously without changing the form of f_s (Wegner 1972); or (*b*) the invariance property of the homogeneous scaling equation under the transformation $\lambda_i \to \lambda_i^n$ and $y_i \to y_i^n$ where n is any positive integer, is also assumed to be valid for n real and large. Another argument (Wilson, private communication) that $C_n = 0$ for $n \neq 0$ follows from the observation that the period depends on λ_1 which is a function of L for fixed values of the critical exponent α_1 while the free energy should be independent of L. The contribution of the self-energy to the Fourier coefficient $C_0(\xi)$ (equations (13)–(14)) can also be obtained directly from the series for the singular part of the free energy (equation (9)) by replacing the sum over integers n by an integral over real values of n, that is, the first step of an Euler–Maclaurin expansion, and a similar approximation leads to the expression for $C_0(\zeta)$ at $\zeta = 0$ given by Nauenberg and Nienhuis (1974). Less singular corrections to scaling can be obtained by keeping also the irrelevant variables ζ_j, $j \neq 1, 2$, and the nonlinear dependence of ζ_j on K_α (Wegner 1972).

† A periodic dependence has also been discussed by Jona-Lasinio *The Renormalization Group: A Probabilistic View* (preprint, University of Padova, July 1974).

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